P 0.1. *If a, b, c are positive real numbers such that $a + b + c = 3$, then*

$$\sqrt{a^3 + 3b} + \sqrt{b^3 + 3c} + \sqrt{c^3 + 3a} \geq 6.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$(a^3 + 3b)(a + 3b) \geq (a^2 + 3b)^2.$$

Thus, it suffices to show that

$$\sum \frac{a^2 + 3b}{\sqrt{a + 3b}} \geq 6.$$

By Hölder's inequality, we have

$$\left(\sum \frac{a^2 + 3b}{\sqrt{a + 3b}}\right)^2 \left[\sum (a^2 + 3b)(a + 3b)\right] \geq \left[\sum (a^2 + 3b)\right]^3 = \left(\sum a^2 + 9\right)^3.$$

Therefore, it is enough to show that

$$\left(\sum a^2 + 9\right)^3 \geq 36 \sum (a^2 + 3b)(a + 3b).$$

Let $p = a + b + c = 3$ and $q = ab + bc + ca$, $q \leq 3$. We have

$$\sum a^2 + 9 = p^2 - 2q + 9 = 2(9 - q),$$

$$\sum (a^2 + 3b)(a + 3b) = \sum a^3 + 3 \sum a^2 b + 9 \sum a^2 + 3 \sum ab$$
$$= (p^3 - 3pq + 3abc) + 3 \sum a^2 b + 9(p^2 - 2q) + 3q$$
$$= 108 - 24q + 3(abc + \sum a^2 b).$$

Since $abc + \sum a^2 b \leq 4$, we get

$$\sum (a^2 + 3b)(a + 3b) \leq 24(5 - q).$$

Thus, it suffices to show that

$$(9 - q)^3 \geq 108(5 - q),$$

which is equivalent to the obvious inequality

$$(3 - q)^2 (21 - q) \geq 0.$$

The equality holds for $a = b = c = 1$. $\qquad \square$